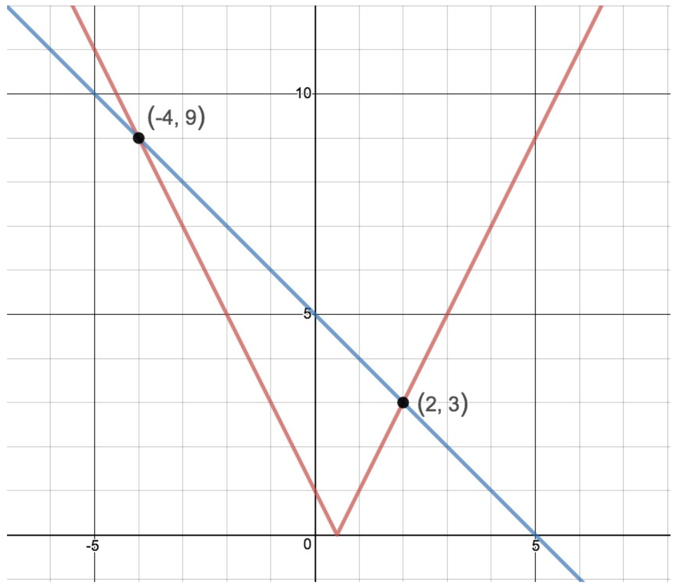
# MATH 1650: GRAPHICAL METHODS TO SOLVE EQUATIONS AND INEQUALITIES

In this section, we discuss methods of using graphs to help us solve equations and inequalities.

**EXAMPLE:** Below are graphed $f(x) = |2x - 1|$ and $g(x) = 5 - x$.




- Which graph is which? How do you know?

  The graph of $f$ is the '$\vee$' shape, since it's an absolute value function.

  The graph of $g$ is the line, since it's a linear function.

- $f(-4) = |2(-4) - 1| = 9$ and $g(-4) = 5 - (-4) = 9$. $f(-4) = g(-4)$

  Geometrically, this means the graphs **intersect** at $(-4, 9)$.

- $f(2) = |2(2) - 1| = 3$ and $g(2) = 5 - 2 = 3$. $f(2) = g(2)$.

  Geometrically, this means the graphs **intersect** at $(2, 3)$.

- $f(0) = |2(0) - 1| = 1$ and $g(0) = 5 - 0 = 5$. $f(0) < g(0)$.

  Geometrically, this means the graph of $f$ is **below** the graph of $g$ at when $x = 0$.

- $f(4) = |2(4) - 1| = 7$ and $g(4) = 5 - 4 = 1$. $f(0) > g(0)$.

  Geometrically, this means the graph of $f$ is **above** the graph of $g$ at when $x = 0$.

**EXAMPLE:** (Continued):

- How does the graph above tell you there are only **two** solutions to $|2x - 1| = 5 - x$? What are they?

  The graphs of $f(x) = |2x - 1|$ and $g(x) = 5 - x$ only intersect twice. $|2x - 1| = 5 - x$ when $x = -4$ and $x = 2$.

- Use the graph above to solve $|2x - 1| < 5 - x$. Write your answer in interval notation.

  The graph of $f(x) = |2x - 1|$ is **below** the graph of $g(x) = 5 - x$ when $-4 < x < 2$

  So the solution to $|2x - 1| < 5 - x$ is $(-4, 2)$.

- Use the graph above to solve $|2x - 1| > 5 - x$. Write your answer in interval notation.

  The graph of $f(x) = |2x - 1|$ is **above** the graph of $g(x) = 5 - x$ from $x < -4$ or $x > 2$.

  So the solution to $|2x - 1| > 5 - x$ is $(-\infty, -4) \cup (2, \infty)$.

- To solve $|2x - 1| \geq 5 - x$, we look for where $f(x) > g(x)$ or where $f(x) = g(x)$.

  That is, where $f(x) = |2x - 1|$ is above $g(x) = 5 - x$ or where the graphs intersect. The answer is: $(-\infty, -4] \cup [2, \infty)$

**EXAMPLE:** Suppose we wish to solve $|5x-3| \leq 6-2x$.

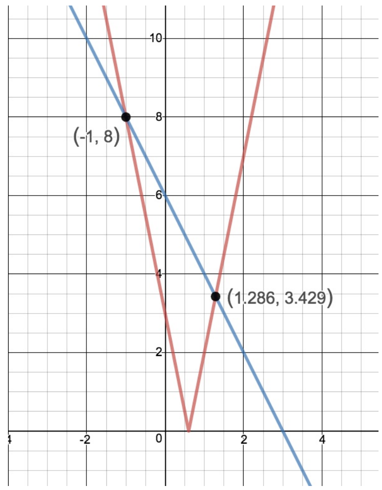
- To solve $|5x-3| = 6-2x$, we remove the absolute values to get: $5x-3 = 6-2x$ or $5x-3 = -(6-2x)$.

  Solving $5x-3 = 6-2x$ gives $7x = 9$ or $x = \dfrac{9}{7}$.

  Solving $5x-3 = -(6-2x)$ gives $5x-3 = -6+2x$ or $3x = -3$ so $x = -1$.

- The $x$-coordinates of the intersection points correspond to $x = -1$ and $x = \dfrac{9}{7}$.

  Note: We get a decimal approximation for $x = \dfrac{9}{7}$.

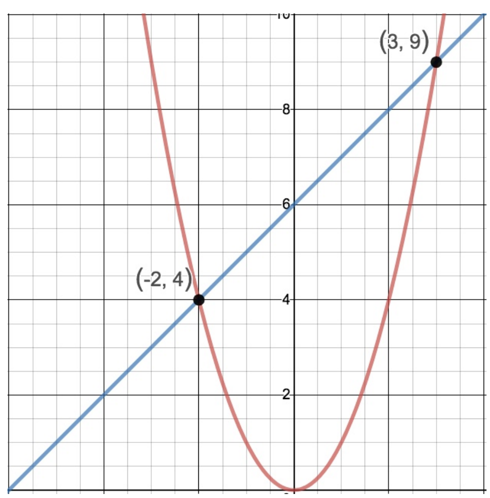



- The solution to $|5x-3| \leq 6-2x$ is where the graph of $f(x) = |5x-3|$ intersects or is below the graph of $g(x) = 6-2x$.

  The answer is $\left[-1, \dfrac{9}{7}\right]$.

**EXAMPLE:** Use the process above to solve $x^2 < x+6$.

- **STEP 1:** Solve $x^2 = x+6$. We rewrite this as $x^2 - x - 6 = 0$ and factor: $(x+2)(x-3) = 0$ so $x = -2$ or $x = 3$.
- **STEP 2:** Graph $f(x) = x^2$ and $g(x) = x+6$. Check the intersection points match with your solutions from STEP 1.




- **STEP 3:** The solution to $x^2 < x+6$ is where the graph of $f(x) = x^2$ is **below** the graph of $g(x) = x+6$.

  From the graph, we see the parabola is below the line between $x = -2$ and $x = 3$.

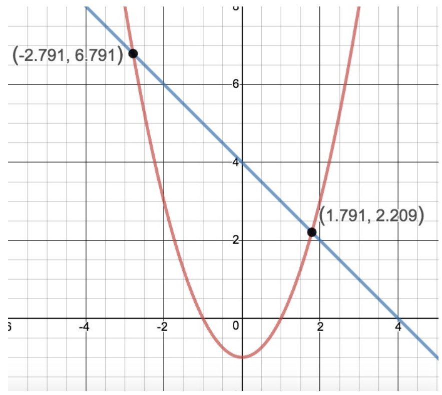
  Using interval notation, our answer is: $(-2, 3)$.

**EXAMPLE:** Solve $x^2 - 1 \leq 4 - x$.

- **STEP 1:** To solve $x^2 - 1 = 4 - x$, we rewrite this as: $x^2 + x - 5 = 0$. Using the quadratic formula, we get:

$$x = \frac{-1 \pm \sqrt{(1)^2 - 4(1)(-5)}}{2(1)} = \frac{-1 \pm \sqrt{21}}{2}$$

- **STEP 2:** Graph $f(x) = x^2 - 1$ and $g(x) = 4 - x$ to check intersection points.




**NOTE:** We are getting decimal approximations for $x = \dfrac{-1 - \sqrt{21}}{2}$ and $x = \dfrac{-1 + \sqrt{21}}{2}$.

- **STEP 3:**

The solution to $x^2 - 1 \leq 4 - x$ is where the graph of $f(x) = x^2 - 1$ intersects or is **below** the graph of $g(x) = 4 - x$.

Our answer is $\left[\dfrac{-1 - \sqrt{21}}{2}, \dfrac{-1 + \sqrt{21}}{2}\right]$.

**CHALLENGE!** Solve $x^2 - 2 > |2x + 1|$:

- **STEP 1:** To solve $x^2 - 2 = |2x + 1|$, we drop the absolute values to get: $2x + 1 = x^2 - 2$ or $2x + 1 = -\left(x^2 - 2\right)$.

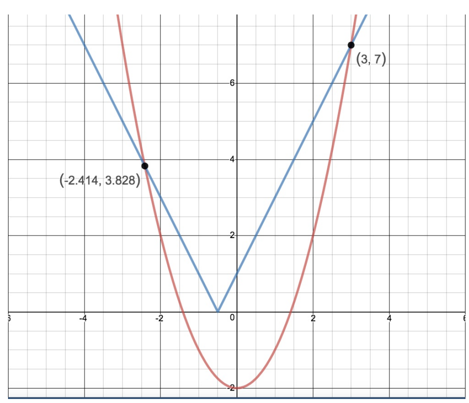
Rewriting $2x + 1 = x^2 - 2$, we get $x^2 - 2x - 3 = 0$ or $(x + 1)(x - 3) = 0$. Our answer here are $x = -1$ or $x = 3$.

Rewriting $2x + 1 = -\left(x^2 - 2\right)$, we get $2x + 1 = -x^2 + 2$ so $x^2 + 2x - 1 = 0$. The quadratic formula gives:

$$x = \frac{-2 \pm \sqrt{(2)^2 - 4(1)(-1)}}{2} = \frac{-2 \pm \sqrt{4 + 4}}{2} = \frac{-2 \pm \sqrt{8}}{2} = \frac{-2 \pm 2\sqrt{2}}{2} = -1 \pm \sqrt{2}$$

- **STEP 2:** From the graph, we see only $x = -1 - \sqrt{2}$ and $x = 3$ are solutions to the equation.

**NOTE:** Recall this means $x = -1$ and $x = -1 + \sqrt{2}$ are called **extraneous** solutions.

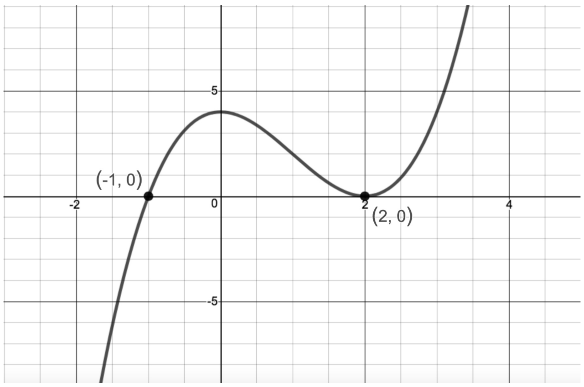



- The solution to $x^2 - 2 > |2x + 1|$ is where the graph of $f(x) = x^2 - 2$ is **above** the graph of $g(x) = |2x + 1|$.

So our answer is: $(-\infty, -1 - \sqrt{2}) \cup (3, \infty)$.

## SOLVING INEQUALITIES WITH SIGN DIAGRAMS

**EXAMPLE:** Use the graph below on the left of $y = f(x)$ to make a sign diagram for $f(x)$.




The graph of $y = f(x)$

$$\begin{array}{ccccccccc} & (-) & 0 & (+) & 0 & (+) & f(x) \\ -\infty \leftarrow & & \vert & & \vert & & \rightarrow \infty \\ & & -1 & & 2 & & x \end{array}$$

A Sign Diagram for $f(x)$

**EXAMPLE:** Follow the steps outlined above to make a Sign Diagram for $f(x) = 4 - x^2$.

- **STEP 1:** Solve $f(x) = 0$: We solve $4 - x^2 = 0$ by extracting square roots: $x^2 = 4$ so $x = \pm\sqrt{4} = \pm 2$.
- **STEP 2:** $x = -2$ and $x = 2$ divide the real number line into three intervals: $x < -2$, $-2 < x < 2$, and $x > 2$.

  For $x < -2$: we choose $x = -3$ as our test value and find $f(-3) = 4 - (-3)^2 = -5$, so $f(x)$ is $(-)$ here.

  For $-2 < x < 2$: we choose $x = 0$ as our test value and find $f(0) = 4 - (0)^2 = 4$, so $f(x)$ is $(+)$ here.

  For $x > 2$: we choose $x = 3$ as our test value and find $f(3) = 4 - (3)^2 = -5$, so $f(x)$ is $(-)$ here.
- **STEP 3:** Our Sign Diagram is below:

$$\begin{array}{ccccccccc} & (-) & 0 & (+) & 0 & (-) & f(x) \\ -\infty \leftarrow & & \vert & & \vert & & \rightarrow \infty \\ & \uparrow & -2 & \uparrow & 2 & \uparrow & x \\ & -3 & & 0 & & 3 & \end{array}$$

**EXAMPLE:** Solve the following inequalities using a Sign Diagram. Check your answers graphically.

- $x^2 \geq x + 12$: Rewriting as $x^2 - x - 12 \geq 0$, we identify $f(x) = x^2 - x - 12$ and make a sign diagram.

$$\begin{array}{ccccccccc} & (+) & 0 & (-) & 0 & (+) & f(x) \\ -\infty \leftarrow & & \vert & & \vert & & \rightarrow \infty \\ & \uparrow & -3 & \uparrow & 4 & \uparrow & x \\ & -4 & & 0 & & 5 & \end{array}$$

  Since we are solving $f(x) = x^2 - x - 12 \geq 0$, we are looking for where $f(x)$ is either $(+)$ or $0$.

  Our solution is: $(-\infty, -3] \cup [4, \infty)$.

- $x^2 - 4x < 1$: Rewriting as $x^2 - 4x - 1 < 0$, we identify $f(x) = x^2 - 4x - 1$ and make a sign diagram.

$$\begin{array}{ccccccccc} & (+) & 0 & (-) & 0 & (+) & f(x) \\ -\infty \leftarrow & & \vert & & \vert & & \rightarrow \infty \\ & \uparrow & 2-\sqrt{5} & \uparrow & 2+\sqrt{5} & \uparrow & x \\ & -1 & & 2 & & 5 & \end{array}$$

  Since we are solving $f(x) = x^2 - 4x - 1 < 0$, we are looking for where $f(x)$ is $(-)$.

  Our solution is: $(2 - \sqrt{5}, 2 + \sqrt{5})$.

- $x(3 - x) < 3(x + 2)$: Rewriting, we get $3x - x^2 < 3x + 6$ or $x^2 + 6 > 0$.

  $x^2 + 6 > 0$ is true for all values of $x$, so the answer is: $(-\infty, \infty)$.